Limit of a function (examples I part)

1) Figure out the limits the following functions:

a) $\lim_{x\to 2} 2x =$

b) $\lim_{x\to 1} \frac{x+2}{3x} =$

c) $\lim_{x\to 5} \frac{10}{x-5} =$

d) $\lim_{x\to -3} \frac{x+3}{7} =$

Solution:

What to do?

Where we see x change it with the value of which he tends!

a) $\lim_{x\to 2} 2x = 2\cdot 2 = 4$

b) $\lim_{x\to 1} \frac{x+2}{3x} = \frac{1+2}{3\cdot 1} = \frac{3}{3} = 1$

c) $\lim_{x\to 5} \frac{10}{x-5} = \frac{10}{5-5} = \frac{10}{0} = \infty$

d) $\lim_{x\to -3} \frac{x+3}{7} = \frac{-3+3}{7} = \frac{0}{7} = 0$

2. Find the limits of following functions:

a) $\lim_{x\to\infty} \frac{3x+1}{2x^2-5x+6}$

b) $\lim_{x\to\infty} \frac{2x^3-3x+12}{x^2-5}$

c) $\lim_{x\to\infty} \frac{5x^2-3x+2}{2x^2+4x+1}$

d) $\lim_{x\to\infty} \frac{x^2+1}{3-x^2}$

Solution:

Remember: In this type of task, where $x \to \infty$, function is rational, $\frac{f(x)}{Q(x)}$,and there is no root, ln, sin or other functions use the following conclusions:

i) If the highest level up to dividend is greater than the highest level down to divisor, **solution is** ∞ .
ii) If it is the highest level below greater than the highest level up, the solution is 0.
iii) If the highest levels are the same, the solution is quotient coefficients in front of the largest degree.

$$\text{a)} \lim_{x\to\infty}\frac{3x+1}{2x^2-5x+6}=0 \quad \text{(conclusion ii)}$$

$$\text{b)} \lim_{x\to\infty}\frac{2x^3-3x+12}{x^2-5}=\infty \quad \text{(conclusion i)}$$

$$\text{c)} \lim_{x\to\infty}\frac{5x^2-3x+2}{2x^2+4x+1}=\frac{5}{2} \quad \text{(conclusion iii)}$$

$$\text{d)} \lim_{x\to\infty}\frac{x^2+1}{3-x^2}=\frac{1}{-1}=-1 \quad \text{(conclusion iii)}$$

3. Find the limits of following functions:

$$\text{a)} \lim_{x\to 2}\frac{x^2-4}{x-2};$$

$$\text{b)} \lim_{x\to 1}\frac{x^2+6x-7}{x^2-5x+4};$$

$$\text{c)} \lim_{x\to -3}\frac{x^4-6x^2-27}{x^3+3x^2+x+3};$$

Solution:

This type of problems we deal with transformation of algebraic expressions:

i) ax + ay +az = a(x+y+z) and xa+ya+za=(x+y+z)a law of distributivity

ii) Formulas:

$$A^2-B^2=(A-B)\cdot(A+B)$$

$$A^2\pm 2AB+B^2=(A\pm B)^2$$

$$A^3-B^3=(A-B)(A^2+AB+B^2)$$

$$A^3+B^3=(A+B)(A^2-AB+B^2)$$

$$A^3+3A^2B+3AB^2+B^3=(A+B)^3$$

$$A^3-3A^2B+3AB^2-B^3=(A-B)^3$$

iii) Square equations apart on the factors, with $ax^2+bx+c=a(x-x_1)(x-x_2)$

a)

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{(x-2)(x+2)}{x-2}=\lim_{x\to 2}(x+2)=2+2=4$$

b)

$$\lim_{x\to 1}\frac{x^2+6x-7}{x^2-5x+4}\Rightarrow \quad \text{We will use : } \quad ax^2+bx+c=a(x-x_1)(x-x_2)$$

$x^2+6x-7=0$	$x^2-5x+4=0$
$x_{1,2}=\dfrac{-6\pm\sqrt{64}}{2}$	$x_{1,2}=\dfrac{5\pm\sqrt{9}}{2}$
$x_{1,2}=\dfrac{-6\pm 8}{2}$	$x_{1,2}=\dfrac{5\pm 3}{2}$
$x_1=1$	$x_1=4$
$x_2=-7$	$x_2=1$
$x^2+6x-7=(x-1)(x+7)$	$x^2-5x+4=(x-4)(x-1)$

$$\lim_{x\to 1}\frac{x^2+6x-7}{x^2-5x+4}=\lim_{x\to 1}\frac{(x-1)(x+7)}{(x-4)(x-1)}=\lim_{x\to 1}\frac{x+7}{x-4}=\frac{1+7}{1-4}=\frac{8}{-3}$$

c) $\displaystyle\lim_{x\to -3}\frac{x^4-6x^2-27}{x^3+3x^2+x+3}\Rightarrow$ This will separate and do…

$x^4-6x^2-27=0\rightarrow$ replacement $x^2=t$

$t^2-6t-27=0$

$t_{1,2}=\dfrac{6\pm 12}{2}$

$t_1=9$

$t_2=-3$

$x^4-6x^2-27=(x^2-9)(x^2+3)=(x-3)(x+3)(x^2+3)$

$x^3+3x^2+x+3=x^2(x+3)+1(x+3)=(x+3)(x^2+1)$

Let's go back now to task:

$$\lim_{x\to -3}\frac{x^4-6x^2-27}{x^3+3x^2+x+3}=\lim_{x\to -3}\frac{(x-3)(x+3)(x^2+3)}{(x+3)(x^2+1)}=\lim_{x\to -3}\frac{(x-3)(x^2+3)}{(x^2+1)}=$$

$$=\lim_{x\to -3}\frac{(-3-3)(9+3)}{(9+1)}=\frac{-6\cdot 12}{10}=-\frac{72}{10}=-\frac{36}{5}$$

l'Hôpital's rule

This type of task is (if possible and if you know derivates) the best work over **l'Hôpital's rule.**

a) $\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{2x}{1}=2*2=4$

b) $\lim_{x\to 1}\frac{x^2+6x-7}{x^2-5x+4}=\lim_{x\to 1}\frac{2x+6}{2x-5}=\frac{2\cdot 1+6}{2\cdot 1-5}=\frac{8}{-3}$

c) $\lim_{x\to -3}\frac{x^4-6x^2-27}{x^3+3x^2+x+3}=\lim_{x\to -3}\frac{4x^3-12x}{3x^2+6x+1}=\frac{4\cdot(-3)^3-12\cdot(-3)}{3\cdot(-3)^3+6\cdot(-3)+1}==\frac{-4\cdot 27+36}{27-18+1}=\frac{-72}{10}=-\frac{36}{5}$

Much faster and easier, does not it?

4) Determine the following limits:

a) $\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{4x}$;

b) $\lim_{x\to 3}\frac{x-3}{\sqrt{x+1}-2}$;

c) $\lim_{x\to 0}\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+16}-4}$;

This is a new type of task, with roots. The idea is to make rationalization. Use formula:

$$A^2-B^2=(A-B)\cdot(A+B)$$

Solution:

a)

$$\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{4x}=\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{4x}\cdot\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}=$$

$$=\lim_{x\to 0}\frac{\sqrt{1+x}^2-\sqrt{1-x}^2}{4x\left(\sqrt{1+x}+\sqrt{1-x}\right)}=\lim_{x\to 0}\frac{1+x-1+x}{4x\left(\sqrt{1+x}+\sqrt{1-x}\right)}=$$

$$=\lim_{x\to 0}\frac{2x}{4x\left(\sqrt{1+x}+\sqrt{1-x}\right)}=\lim_{x\to 0}\frac{1}{2\left(\sqrt{1+x}+\sqrt{1-x}\right)}=$$

$$=\frac{1}{2\left(\sqrt{1+0}+\sqrt{1-0}\right)}=\frac{1}{2\cdot 2}=\frac{1}{4}$$

b)

$$\lim_{x\to 3}\frac{x-3}{\sqrt{x+1}-2}=\lim_{x\to 3}\frac{x-3}{\sqrt{x+1}-2}\cdot\frac{\sqrt{x+1}+2}{\sqrt{x+1}+2}=$$
$$=\lim_{x\to 3}\frac{(x-3)\left(\sqrt{x+1}+2\right)}{\sqrt{x+1}^2-2^2}=\lim_{x\to 3}\frac{(x-3)\left(\sqrt{x+1}+2\right)}{x+1-4}=\lim_{x\to 3}(\sqrt{x+1}+2)$$
$$=\sqrt{3+1}+2=2+2=4$$

c)

$$\lim_{x\to 0}\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+16}-4}$$ You have to make double rationalization.

$$\lim_{x\to 0}\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+16}-4}\cdot\frac{\sqrt{x^2+1}+1}{\sqrt{x^2+1}+1}\cdot\frac{\sqrt{x^2+16}+4}{\sqrt{x^2+16}+4}=$$
$$=\lim_{x\to 0}\frac{(x^2+1-1)\left(\sqrt{x^2+16}+4\right)}{(x^2+16-16)\left(\sqrt{x^2+1}+1\right)}=\lim_{x\to 0}\frac{x^2\left(\sqrt{x^2+16}+4\right)}{x^2\left(\sqrt{x^2+1}+1\right)}=$$
$$=\lim_{x\to 0}\frac{\sqrt{x^2+16}+4}{\sqrt{x^2+1}+1}=\frac{\sqrt{0+16}+4}{\sqrt{0+1}+1}=\frac{8}{2}=4$$

5) Determine the following limits:

a) $\lim_{x\to 0}\frac{\sqrt[3]{1+x^2}-1}{x^2}$;

b) $\lim_{x\to 0}\frac{\sqrt[3]{x}-1}{\sqrt{x}-1}$;

Solution:

Watch out: When we have a third roots, we must use the formulas:

$$(A-B)(A^2+AB+B^2)=A^3-B^3$$
$$(A+B)(A^2-AB+B^2)=A^3+B^3$$

a)

$$\lim_{x\to 0}\frac{\sqrt[3]{1+x^2}-1}{x^2}=\lim_{x\to 0}\frac{\sqrt[3]{1+x^2}-1}{x^2}\cdot\frac{\sqrt[3]{(1+x^2)^2}+\sqrt[3]{1+x^2}+1}{\sqrt[3]{(1+x^2)^2}+\sqrt[3]{1+x^2}+1}=$$
$$=\lim_{x\to 0}\frac{\sqrt[3]{1+x^2}^3-1^3}{x^2\left[\sqrt[3]{(1+x^2)^2}+\sqrt[3]{1+x^2}+1\right]}=$$
$$=\lim_{x\to 0}\frac{1+x^2-1}{x^2\left[\sqrt[3]{(1+x^2)^2}+\sqrt[3]{1+x^2}+1\right]}=$$
$$=\lim_{x\to 0}\frac{1}{\sqrt[3]{(1+x^2)^2}+\sqrt[3]{1+x^2}+1}=\frac{1}{\sqrt[3]{(1+0)^2}+\sqrt[3]{1+0}+1}=\frac{1}{3}$$

b)

$$\lim_{x\to 0}\frac{\sqrt[3]{x}-1}{\sqrt{x}-1}=$$ **Look out:** you have to make double rationalization

$$\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{\sqrt{x}-1}\cdot\frac{\sqrt[3]{x^2}+\sqrt[3]{x}+1}{\sqrt[3]{x^2}+\sqrt[3]{x}+1}\cdot\frac{\sqrt{x}+1}{\sqrt{x}+1}=$$

$$=\lim_{x\to 1}\frac{\left(\sqrt[3]{x}^3-1^3\right)\left(\sqrt{x}+1\right)}{\left(\sqrt{x}^2-1^2\right)\left(\sqrt[3]{x^2}+\sqrt[3]{x}+1\right)}=\lim_{x\to 1}\frac{(x-1)\left(\sqrt{x}+1\right)}{(x-1)\left(\sqrt[3]{x^2}+\sqrt[3]{x}+1\right)}=$$

$$=\lim_{x\to 1}\frac{\sqrt{x}+1}{\sqrt[3]{x^2}+\sqrt[3]{x}+1}=\frac{1+1}{1+1+1}=\frac{2}{3}$$